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Cardinality constraints in semantic data models

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Abstract

Constraints are central to the notion of a semantic data model. How well a model captures constraints affects its power and viability as a semantic data model. Cardinality constraints are an important subclass of general constraints. In this paper we provide formal definitions for cardinality constraints of several semantic models, as described in the literature. We construct a partial ordering of these constraints that shows the relative power expressed by each cardinality constraint. We discuss our results and offer possible extensions to contemporary cardinality constraint definitions. Our contributions include a collection and formal definition of existing cardinality constraints, a partial ordering of this set, and recommendations for cardinality constraint mechanisms in semantic data models.

Keywords. Cardinality; cardinality constraints; semantic data models; constraints in semantic data models.

1. Introduction

A semantic data model captures and expresses semantic information in a way that is accessible to humans, yet is precise and formal [3, 12, 13, 20, 25]. Semantic models allow users to organize basic modeling constructs and establish constraints over these basic constructs. The Entity-Relationship (ER) Model, for example, provides us with entity sets, relationship sets, and attributes as basic modeling constructs, and allows us to express constraints such as "binary relationship set r is 1-to-1" [4].

Although semantic models support various basic modeling constructs and several categories of constaints, we focus in this paper only on constraints and in particular only on cardinality constraints. Cardinality constraints are of interest because they capture semantic detail about the structure of sets of relationships or entities in data model instances.

Cardinality refers to the number of elements in a mathematical set [1]. Thus, a *cardinality constraint* is a constraint that restricts the number of elements in a set. For example, if C is a set of computers and D is a set of disk drives, we may require that all computers in C connect to at least one disk drive in D. *Figure 1(a)* diagrams a legal instance for this constraint, whereas *Fig. 1(b)* shows an instance that violates the constraint. If we let r represent the binary set of connections from C to D, this cardinality constraint can be written as $\forall c \in C(|\{d : d \in D \wedge \langle c, d\rangle \in r\}| \geq 1)$. Here, and throughout the paper, $\{x : P(x)\}$ denotes a set of x's that satisfies predicate P over x, and $|S|$ denotes the cardinality of set S.

Semantic data models have provided various ways to specify cardinality constraints. In this paper we examine several cardinality constraints in existing semantic models, and we present formal definitions for these constraints. We present definitions for relationship-set cardinality

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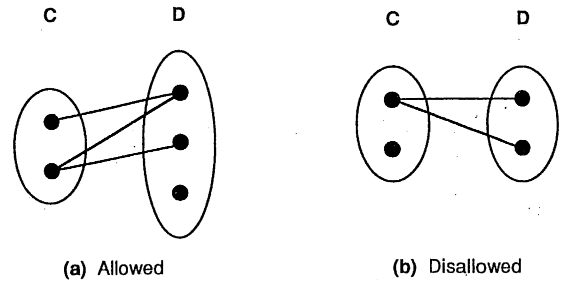

Fig. 1. Mapping under a cardinality constraint.

constraints in Section 2. In Section 3 we give a partial ordering of the cardinality constraints for n-ary relationships and observe how the partial ordering changes when we restrict relationships to be binary. We present definitions for entity-set cardinality constraints in Section 4, and discuss how entity-set and relationship-set cardinality constraints together modify the partial orderings of Section 3. We discuss our results in Section 5, and also present some extensions to existing cardinality constraint definitions. Finally, we give our conclusions in Section 6.

2. Relationship-set cardinality constraints

We present formal definitions for cardinality constraints for the following semantic data models: Semantic Binary Data Model (1974), Entity-Relationship Model (1976), Structural Model (1979), Semantic Association Model (1980), Semantic Database Model (1981), Nijssen's Information Analysis Methodology (1982) and its Binary Relationship Model (1983), Enriched Entity-Relationship Model (1983), Entity-Category-Relationship Model (1985), Extended Entity-Relationship Model (1986), Iris (1987), Object Modeling Technique (1991), and Object-oriented Systems Analysis (1992). We present these definitions in their approximate order of appearance in the literature. Several semantic data models are not included in this list, but after extensive study of the literature we believe that those listed represent the different kinds of cardinality constraints in use today.

To obtain uniformity in describing sets of objects and sets of relationships among objects, we base our terminology on the ER model. Thus, we give definitions for all models in terms of entity sets and relationship sets. Furthermore, we assume that an entity set is a set of unique entities, independent of any attributes, and we assume that relationship sets are tuples of entities aggregated from their associated entity sets. We also assume that referential integrity holds. For this paper, we ignore recursive relationship sets. An extension of this discussion to include recursive relationship sets is conceptually straightforward, but notationally cumbersome.

For our formal definitions, we use predicate calculus with elements of relational algebra. We use angle brackets $\langle \cdots \rangle$ to denote a tuple in a relationship set, and we use π to denote a projection of a relationship set onto a subset of its associated entity sets. We also consider an element a of an entity set E to be equal to $\langle a \rangle$ taken from a projection of a relationship set onto E. **N** denotes the set of natural numbers $\{0, 1, 2, \ldots\}$, whereas **P** denotes the set of positive integers $\{1, 2, 3, \ldots\}$.

2.1. Semantic Binary Data Model

One of the earliest semantic models is the Semantic Binary Data Model (SBDM), introduced by Abrial [2]. This model is considered a minimalist model, since it defines a relatively small set of basic constructs, including only binary (as opposed to n-ary) relationships. The fundamental constructs of SBDM include objects, categories, and binary relationships. A relationship (**rel**) is specified by listing two categories and two access functions (**afn**'s) that define the relationship. For example, if *PERSON* and *NUMBER* were categories, and we wished to specify an age relationship r, we would write

$$r = \mathbf{rel}(PERSON, NUMBER, AGE = \mathbf{afn}(1, 1), PERSONOFAGE = \mathbf{afn}(0, \infty)) .$$

The two access functions *AGE* and *PERSONOFAGE* are mappings that are inverses of each other.

Each access function has an associated cardinality constraint denoted by a pair of numbers (a *minimum* and a *maximum cardinal*) in parentheses. The meaning of a cardinality constraint (min, max) is that the corresponding access function maps an entity from one set to at least min and at most max entities of the other set. Given a binary relationship $r = \mathbf{rel}(E_1, E_2, r_1 = \mathbf{afn}(min_1, max_1), r_2 = \mathbf{afn}(min_2, max_2))$, between entity sets E_1 and E_2, we can formally define the SBDM cardinality constraint as

$$\forall e_1 \in E_1(min_1 \leq |\{t \in r_1 : t(E_1) = e_1\}| \leq max_1) . \qquad \text{(SBDM)}$$

A similar formula defines the cardinality constraint on the second access function. The cardinals may take on values in the set $\mathbf{N}$, and maximum cardinals may also have the value ∞, which represents no upper limit.

A variety of semantics are associated with SBDM cardinality constraints. For example, a minimum cardinal of 1 corresponds to a total mapping, and a maximum cardinal of 1 restricts the mapping to be functional [2].

Finally, we observe that although SBDM does have a graphical representation for categories and relationships, no provision is made for the specification of cardinality constraints as part of the graphical representation of a model instance. If we wish to specify SBDM cardinality constraints, we must write them down using a separate declarative language.

2.2. Entity-Relationship Model

Chen introduced the Entity-Relationship (ER) model in 1976 [4]. Cardinality constraints in the ER model are restrictions on the mapping of relationship sets. *Figure 2* shows three kinds of mappings Chen defined for binary relationship sets: 1: 1, 1: M, and M: N. Let A and B be the entity sets associated by relationship set r. The M: N mapping in *Fig. 2(a)* means that each entity of A may be associated with multiple entities in B and vice versa. An M: N mapping imposes no constraint on cardinality. The mapping 1: M in *Fig. 2(b)* means that each entity in A can have multiple related entities in B, but each entity in B can be related to at most one entity in set A. This is defined formally as follows:

$$\forall \langle b \rangle \in \pi_B r(|\{t \in r : t(B) = b\}| = 1) . \qquad \text{(a)}$$

Note the use of projection (π). This indicates that the constraint only applies to entities that participate in r; thus the mapping may be partial. The 1: 1 mapping in *Fig. 2(c)* means that

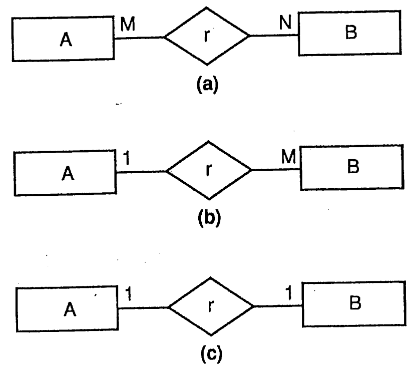

Fig. 2. ER constraints for binary relationship sets.

each entity of A can be related to at most one B and vice versa. Formally, this is

$$\forall\langle a\rangle \in \pi_A r(|\{t \in r: t(A)=a\}|=1) \wedge$$
$$\forall\langle b\rangle \in \pi_B r(|\{t \in r: t(B)=b\}|=1) . \qquad \text{(b)}$$

Observe that (b) is (a) imposed on both entity sets in the relationship set. Again, the constraint cannot require that the mapping be total.

Although Chen used mapping constraints in n-ary relationship sets in his original paper, he did not give definitions or explicit examples of their meaning. We surmise from later writings, particularly Teorey et al. [24], that Chen's meaning of mapping for n-ary relationship sets is that a connection marked by a 1 designates that the composite of keys of all other connected entity sets constitutes a candidate key for the n-ary relationship set. *Figure 3* shows, for example, the four mapping combinations for ternary relationship sets. Letting the entity-set names stand for keys, the *MNP* mapping in *Fig. 3(a)* means that *ABC* is a candidate key; the *MN1* mapping in *Fig. 3(b)* means that *AB* is a candidate key; the *M11* mapping in *Fig. 3(c)* means that both *AB* and *AC* are candidate keys; and the *111* mapping in *Fig. 3(d)* means that *AB*, *AC*, and *BC* are each candidate keys.

We formally define Chen's n-ary mapping as follows. Let E be a set consisting of the n entity sets for relationship set r, and let $E' \subseteq E$ be the set of entity sets from E that are

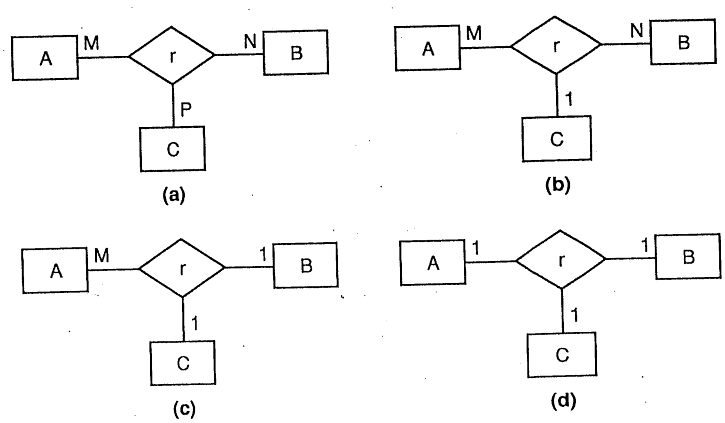

Fig. 3. ER constraints for N-ary relationship sets.

marked by the mapping symbol 1. For each $A \in E'$, letting $E - A = B_1 \ldots B_{n-1}$, the following must hold:

$$\forall \langle b_1, \ldots, b_{n-1} \rangle \in \pi_{B_1 \ldots B_{n-1}} r$$
$$(|\{t \in r : t(B_1) = b_1 \wedge \cdots \wedge t(B_{n-1}) = b_{n-1}\}| = 1) . \qquad \text{(ER)}$$

Note that (a) and (b) are special cases of (ER).

2.3. Structural Model

At Stanford University in the late 1970s, Wiederhold and Elmasri introduced the Structural Model [27, 6, 28]. The Structural Model is based on the Relational Model [5]. Among the differences are that Structural Model relations, which they call classes, are always in Boyce Codd Normal Form; each relation has a ruling part (key) and a dependent part; and several class types are defined that represent real-world objects. These class types are characterized by connections between relation schemas. *Figure 4* illustrates this; K_i, $i = 1$ or $i = 2$, represents the ruling part of relation schema R_i, while $R_i - K_i$ is the dependent part. A *connection* between two relation schemas R_1 and R_2 is defined by two sets $C_1 \subseteq R_1$ and $C_2 \subseteq R_2$ such that the domains of C_1 and C_2 are the same. As shown in *Fig. 4*, R_1 is said to be connected to R_2 through (C_1, C_2). Connections are directed: the box on the end of the line in *Fig. 4* indicates that the direction is from C_1 to C_2. Let r_1 and r_2 be relations on relation schemas R_1 and R_2 respectively. If $t_1 \in r_1$, $t_2 \in r_2$ and $t_1(C_1) = t_2(C_2)$, then an instance of the connection $R_1(C_1, C_2)R_2$ exists between t_1 and t_2. Connections are always binary, but the Structural Model does provide for the aggregation of several connections through a central relation and thus effectively provides n-ary connections.

Connections may have cardinality constraints of the form $m: n$, where m and n are positive integers. The constraint $m: n$ may take the forms 1: 1, 1: m, and $m: n$, corresponding respectively to one-to-one, one-to-many, and many-to-many cardinalities. Integers m and n fix the maximum number of entities (tuples) in one class (relation) that can be related to entities in the other class.

A *dependency constraint* determines whether participation in the connection is mandatory or optional. A *total dependency* makes participation mandatory for both classes. A *partial dependency* makes participation mandatory for all entities of one class in a connection, but allows optional participation for entities in the other class. The specification of *no dependency* allows optional participation for both classes. A connection is undirected. Thus, when a partial dependency is specified, the optional and mandatory sides must be explicitly indicated. The literature on the Structural Model does not indicate precisely how dependen-

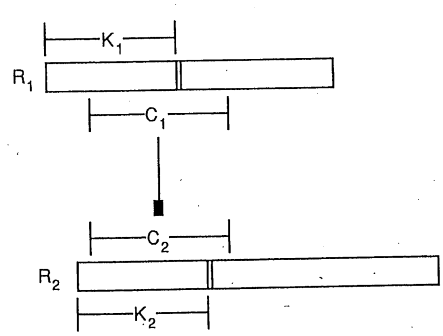

Fig. 4. Structural model inter-relation connection.

cies and cardinality constraints may be written. We presume that a textual description would accompany the graphical description.

Together, cardinality and dependency constraints allow the description of total or partial one-to-one, one-to-many, and many-to-many relationships. To compare the Structural Model with other models, we view tuples in a relation as entities and a connection as a relationship between two entites. Let A and B be entity sets (relation schemas). If A is connected to B via connection r with dependency d, and this connection has a cardinality constraint $m: n$, $m, n \in \mathbf{P} \cup \{\infty\}$, where m applies to A and n applies to B, then the cardinality constraint is defined as follows:

$$\forall \langle a \rangle \in \pi_A r(|\{t \in r: t(A) = a\}| \leq m) \wedge$$
$$\forall \langle b \rangle \in \pi_B r(|\{t \in r: t(B) = b\}| \leq n) . \qquad \text{(SM)}$$

If the dependency constraint d is total, then we have additionally that $\pi_A r = A$ and $\pi_B r = B$. If d specifies that r is partial only with respect to A, then we have $\pi_B r = B$; if r is partial only with respect to B, then $\pi_A r = A$. If d specifies no dependency, then we simply have (SM).

2.4. Semantic Association Model

The Semantic Association Model (SAM) is by Su and Lo [22]. SAM was originally oriented towards business database modeling, but was extended to allow more natural modeling of scientific and statistical databases [23]. In SAM, a concept may be either *atomic* (cannot be decomposed) or *nonatomic*. A nonatomic concept is formed by specifying an *association* of atomic or nonatomic concepts. The various kinds of associations available distinguishes SAM from other models.

Reverting to ER terminology, an entity set in SAM may have single-valued and multi-valued attributes, where the attribute may be total or partial. Mapping cardinality constraints of relationship sets may be $m-n$, $1-m$, or $1-1$, indicating many-to-many, one-to-many, or one-to-one respectively. *Figure 5* illustrates how these relationship sets are represented in

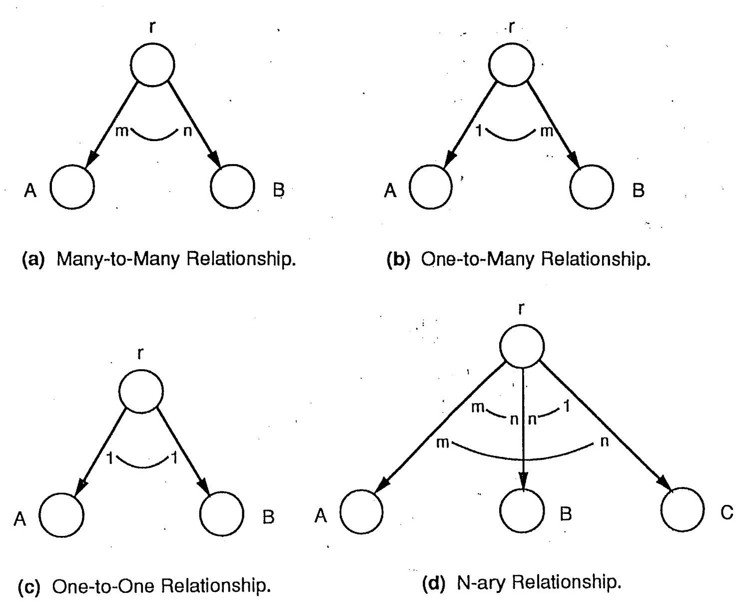

Fig. 5. Semantic association model constraints.

SAM. *Figure 5(a)* shows a many-to-many relationship set r between entity sets A and B, whereas *Figs. 5(b)* and *5(c)* show one-to-many and one-to-one relationship sets, respectively. *Figures 5(a)–5(c)* show binary relationship sets. SAM allows n-ary relationship sets, $n \geq 2$, as in *Fig. 5(d)*, where r is a ternary relationship set among entity sets A, B, and C.

In addition to mapping constraints, SAM allows for the specification of candidate keys for a relationship set. Such a key may involve any number of attributes, but only one key per relationship set may be specified. Thus, there are two definitions for SAM cardinality constraints: mapping constraints and candidate key constraints.

Let A and B be entity sets associated by relationship set r. A many-to-many mapping, $m - n$, shown in *Fig. 5(a)*, imposes no constraint on the mapping cardinality. A $1 - m$ mapping from A to B, as diagrammed in *Fig. 5(b)*, means that each entity in B can be related to at most one entity in A, but an entity in B may be related to many entities in A. We express this formally as follows:

$$\forall \langle b \rangle \in \pi_B r(|\{t \in \pi_{AB} r : t(B) = b\}| = 1) . \tag{c}$$

A $1 - 1$ mapping, shown in *Fig. 5(c)*, imposes the above constraint on both A and B as follows:

$$\begin{aligned} &\forall \langle a \rangle \in \pi_A r(|\{t \in \pi_{AB} r : t(A) = a\}| = 1) \wedge \\ &\forall \langle b \rangle \in \pi_B r(|\{t \in \pi_{AB} r : t(B) = b\}| = 1) . \end{aligned} \tag{d}$$

Notice in the ternary relationship set r of *Fig. 5(d)*, that each mapping constraint is binary. In an n-ary relationship set, mapping constraints can only be applied to binary pairs of entity sets. Thus, (c) and (d) apply equally well to both binary and n-ary relationship sets.

In general, a SAM mapping constraint is a collection of pair-wise mapping constraints. We define this formally as follows. Let r be an n-ary relationship set involving entity sets in E. Since a $1 - 1$ mapping from A to B can be viewed as two $1 - m$ mappings, one from A to B and the other from B to A; and since an $m - n$ mapping imposes no constraint, we assume without loss of generality that all mapping constraints are $1 - m$. For each $1 - m$ mapping constraint on a pair of distinct entity sets $A \in E$ and $B \in E$, the following must hold:

$$\forall \langle b \rangle \in \pi_B r(|\{t \in \pi_{AB} r : t(B) = b\}| = 1) . \tag{SAM}$$

For candidate-key constraints, let r be an n-ary relationship set, and let $\{A_1, A_2, \ldots, A_p\}$ be the candidate key for r, $p \leq n$. Then the following must hold:

$$\begin{aligned} &\forall \langle a_1, \ldots, a_p \rangle \in \pi_{A_1 \ldots A_p} r \\ &(|\{t \in r : t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| = 1) . \end{aligned} \tag{SAM-k}$$

Since there can be at most one candidate key for a relationship set, there is only one such constraint for the relationship set.

2.5. Semantic Database Model

The Semantic Database Model (SDM) has its roots at M.I.T. in the late 1970s [19, 10]. SDM was revised and became well-known in the early 1980s when Hammer and McLeod presented the current version [11]. In SDM, an entity set is called a class, and relationship sets are represented through the specification of attributes. An attribute associates an entity of one class with one or more entities of another class. Attributes may be single-valued (the default) or multivalued, and multivalued attributes may be constrained to associate one entity with between *min* and *max* other entities. Also, an attribute may be required to exhaust the class from which it takes values.

Figure 6 illustrates a sample SDM model fragment. There are three classes: *A's*, *B's*, and *C's*. The class *A's* has three attributes: *R*, *S*, and *T*. *R* establishes a many-to-one relationship from *A's* to *B's*, where each of the *A's* is associated with between 2 and 5 *B's*. *S* and *T* establish one-to-one relationships between *A's* and *C's*, and *A's* and *B's* respectively. SDM attributes are essentially one-way mappings. However, a pair of attributes can be declared as inverses so that two-way mappings can be established. Our example shows a bijection between *B's* and *C's* in the form of inverse attributes *U* and *V*. These mappings constitute a bijection because *U* and *V* both cover their codomains. Finally, *R* and *S* + *T* are identifiers (candidate keys) for the *A's* class. We now proceed to the formal definitions, using the example in *Fig. 6*.

In order to compare SDM with other models, we must transform the concepts slightly. Let *A* and *B* be entity sets (classes), and let *A* have a multivalued attribute *R* that maps entities of *A* to entities of *B*, with cardinality constraints *min* and *max*, (2 and 5 respectively in *Fig. 6*). Let *r* be the set of ordered pairs $\langle a, b \rangle$ such that $a \in A$ and $b \in B$ and attribute *R* maps *a* to *b*. Then the following must hold:

$$\forall \langle a \rangle \in \pi_A r(min \leq |\{t \in r : t(A) = a\}| \leq max) \qquad \text{(SDM)}$$

where $min \in \mathbf{P}$, $max \in \mathbf{P} \cup \{\infty\}$ and $min \leq max$. If *R* is a single-valued attribute, then we have (SDM) where $min = max = 1$. If *R* is required to exhaust *B*, then in addition to (SDM) we have $\pi_B r = B$.

A's
  description: first sample SDM class
  member attributes:
    R
      value class: B's
      multivalued with size between 2 **and** 5
    S
      value class: C's
    T
      value class: B's
  identifiers:
    R
    S + T

B's
  description: second sample SDM class
  member attributes:
    U
      value class: C's
      exhausts value class
      inverse: V

C's
  description: third sample SDM class
  member attributes:
    V
      value class: B's
      exhausts value class
      inverse: U

Fig. 6. SDM constraints.

As mentioned previously, SDM incorporates the concept of candidate key. Attributes may be specified as identifiers, or keys. Recall that the class of A's in *Fig. 6* has two candidate keys: the multivalued attribute R and the composite key $S+T$. The formal definition for candidate keys in SDM is as follows (again we revert to ER terminology). Let r be an n-ary relationship set and let $\{A_1, \ldots, A_p\}$ be a candidate key for r, $p \leq n$, where the A_i's are attributes. Then the following must hold:

$$\forall \langle a_1, \ldots, a_p \rangle \in \pi_{A_1 \ldots A_p} r$$
$$(|\{t \in r \colon t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| = 1) . \qquad \text{(SDM-k)}$$

2.6. Nijssen's Information Analysis Methodology and the Binary Relationship Model

Nijssen's Information Analysis Methodology (NIAM) [26] is an analysis methodology that includes a semantic data model, later called the Binary Relationship Model (BRM) [17, 18]. BRM is a semantic data model in which there are only binary relationships among entities. In BRM, an entity set is called an *object type*, and a relationship set is called a *relationship type*. Entity sets A and B may be associated in a relationship set, written as $A(R_1, R_2)B$, where R_1 and R_2 are the *roles* of A and B respectively. Not all entities in set A need be involved in role R_1. The set of entities from A that appear in the tuples for $A(R_1, R_2)B$ is called the *active set* of A for R_1, written $active_{R_1}(A)$. The union of all active sets for roles of A is denoted $active(A)$. All entities of A must belong to $active(A)$.

BRM has several kinds of cardinality constraints, called cardinality, identifier, total, and uniqueness constraints. The most basic is the *cardinality constraint*, shown in *Fig. 7(a)*. Let r represent $A(R_1, R_2)B$. A cardinality constraint $min..max$ for role R_1 of r is defined formally as

$$\forall \langle a \rangle \in \pi_A r (min \leq |\{t \in r \colon t(A) = a\}| \leq max) \qquad \text{(BRM)}$$

where $min \in \mathbf{P}$, $max \in \mathbf{P}$, $1 \leq min \leq max$, and max may be omitted. If max is not specified,

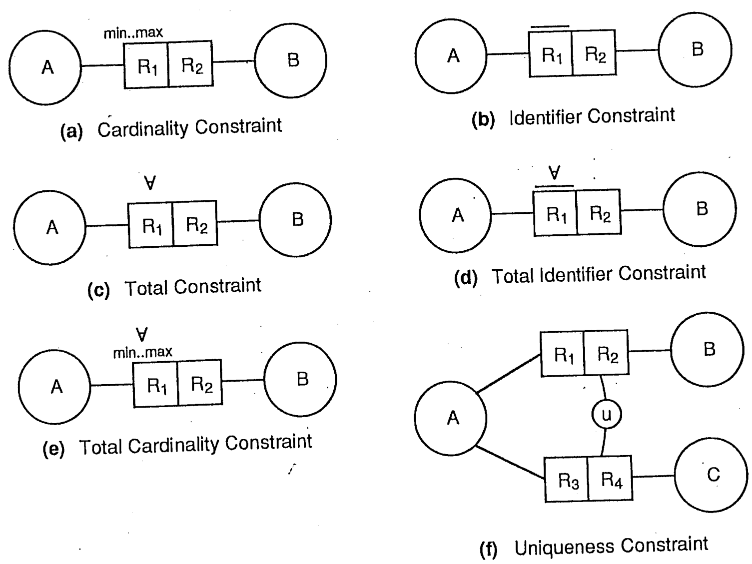

Fig. 7. BRM constraints.

then there is no upper bound on the expression, and thus we may equate *max* with ∞. The cardinality constraint 1 alone (i.e. $min = 1$ and *max* unspecified) expresses no constraint, since any value that appears in the projection of the relationship set must appear at least once. *Figure 7(b)* shows an *identifier constraint*, which is a special case of the cardinality constaint where $min = max = 1$.

A *total constraint* on role R_1, shown in *Fig. 7(c)* specifies that $active(A) = active_{R_1}(A)$, or in our standard terminology, that $\pi_A r = A$. Total constraints may be combined with cardinality constraints, if desired. For example, *Fig. 7(d)* shows a total identifier constraint, and *Fig. 7(e)* shows a total cardinality constraint.

A *uniqueness constraint* is a candidate key. By indicating that a set of object types identifies (is a key for) another object type, *n*-ary relationships can be implied from the binary relationships. *Figure 7(f)* shows a uniqueness constraint; the active sets of *B* and *C* serve as unique identifiers for elements of *A*. In other words, if *ABC* is a relation schema, then *BC* is a candidate key for the schema. The formal definition for candidate keys in BRM is as follows (again we revert to ER terminology). Let *r* be an *n*-ary relationship set and let $\{A_1, \ldots, A_p\}$ be a candidate key for r, $p \leq n$. Then the following must hold:

$$\forall \langle a_1, \ldots, a_p \rangle \in \pi_{A_1 \ldots A_p} r \cdot$$
$$(|\{t \in r : t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| = 1) . \qquad \text{(BRM-k)}$$

2.7. Enriched Entity-Relationship Model

The Enriched Entity-Relationship Model (EER) by Lenzerini and Santucci [15], extends the ER model's fundamental constructs of *entity*, *relationship*, and *attribute*, by adding *subset* and *generalization hierarchies* among entities and also among relationships. EER also includes several integrity constraints not found in the ER model and introduces some new cardinality constraints. These constraints are written as text, and not as part of a graphical diagram.

EER's *relative cardinality constraint* constrains the number of times instances of one set of entities can be related to instances of another set of entities. For example, suppose that entity sets *STUDENT*, *COURSE*, and *GRADE* are related by relationship *ENROLLMENT*. We can indicate that a student may receive at most one grade for a course by specifying a relative cardinality constraint of

$$Cmax[ENROLLMENT(STUDENT/COURSE, GRADE)] = 1 .$$

This can be read as, 'in the relationship set *ENROLLMENT*, a *STUDENT* value may appear with at most one *COURSE*, *GRADE* value pair.'

In addition to a maximum relative cardinality, a minimum relative cardinality constraint may be specified. A minimum cardinality, *Cmin*, may be listed as optional, in which case the minimum is either 0 or *Cmin*. Let *r* be an *n*-ary relationship set, and let $X = \{A_1, \ldots, A_p\}$ and $Y = \{B_1, \ldots, B_q\}$ be disjoint collections of entity sets participating in *r*. The assertion $Cmin[r(X/Y)] = min$ and $Cmax[r(X/Y)] = max$ requires the following:

$$\forall \langle a_1, \ldots, a_p \rangle \cdot \in \pi_{A_1 \ldots A_p} r$$
$$(min \leq |\{t \in \pi_{A_1 \ldots A_p B_1 \ldots B_q} r : t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| \leq max) \qquad \text{(EER-r)}$$

where $min, max > 0$. Note again that *Cmin* or *Cmax* may be undefined, in which case the corresponding inequality is ignored. Furthermore, if *Cmin* is not optional, then $\pi_{A_1 \ldots A_p} r$ must equal the full cross product $A_1 \times \cdots \times A_p$.

Lenzerini and Santucci also defined several important concepts for reasoning about their cardinality constraints. First, a set of cardinality constraints is *consistent* if there exists at least one database state that satisfies every constraint in the set. Second, a set of cardinality constraints is *well-defined* or *sound* if the minimum and maximum cardinalities, and every cardinality in between, are actually attainable for every cardinality constraint in the set.

2.8. *Entity-Category-Relationship Model*

In the Entity-Category-Relationship (ECR) model, Elmasri, Hevner, and Weeldreyer extended the ER model by defining the category concept [7]. A *category* is a classification of entities used for generalization or specialization. Cardinality constraints in the ECR model are based on participation constraints [2, 6, 8]. Given a relationship set r involving entity set A, a *participation constraint* is a pair of integers (*min*, *max*), where each entity in A must participate in r at least *min* and at most *max* times, where $0 \leq min \leq max$ and $1 \leq max$. Also, *max* may be the special symbol ∞, in which case there is no upper limit on participation. The default participation constraint is $(0, \infty)$, representing no constraint on participation.

Figure 8 shows a general relationship set in an ECR diagram. Let $A_1, \ldots, A_n$ be the n entity sets (categories) related by relationship set r. A participation constraint (*min*, *max*) for an entity set A is defined formally as follows:

$$\forall a \in A(min \leq |\{t \in r\colon t(A) = a\}| \leq max) \qquad \text{(ECR)}$$

where $min \in \mathbf{N}$, $max \in \mathbf{P} \cup \{\infty\}$, and $0 \leq min \leq max$. Relationship set r is *total* with respect to entity set A if $min \geq 1$, and *partial* if $min = 0$.

2.9. *Extended Entity-Relationship Model*

In Teorey, Yang, and Fry's Extended Entity-Relationship (XER) model, cardinality constraints are based on mapping cardinalities as in the ER model [24]. However, in addition to specifying a cardinality mapping, each entity set may also be marked as *optional* (partial) or *mandatory* (total) in the relationship set. In an entity set designated as mandatory for a relationship set, all entities must participate in the relationship set.

Figure 9 shows some XER cardinality constraints. Instead of using 1's and letters to mark 'one' and 'many' entity sets, corners of the central polygon are shaded for the 'many' connections. *Figure 9(a)* is thus a 1-many relationship set from A to B. To mark an entity set as optional, a circle is placed on its connecting line. For mandatory participation, the connecting line has no special marking. Thus, in *Fig. 9(a)* entities in A participate optionally while entities in B participate mandatorily in the relationship set. *Fig. 9(b)* shows a ternary relationship between entity sets A, B, and C. The shaded corners mark A and B as many

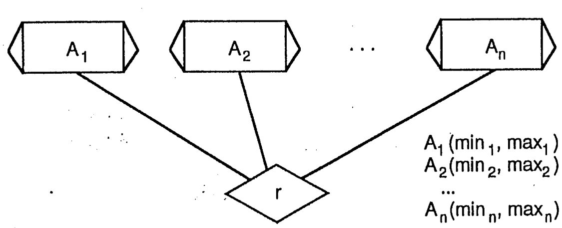

Fig. 8. ECR constraints.

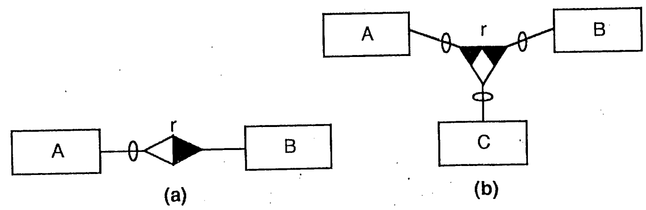

Fig. 9. XER constraints.

sides and the unshaded corner marks C as a 1 side. Participation of all entity sets in this example is optional.

Formally, let E be the n entity sets for relationship set r, and let $E' \subseteq E$ be the entity sets of E attached to unshaded corners. For each $A \in E'$, letting $E - A = B_1 \ldots B_{n-1}$, the following must hold:

$$\forall \langle b_1, \ldots, b_{n-1} \rangle \in \pi_{B_1 \ldots B_{n-1}} r \\ (|\{t \in r : t(B_1) = b_1 \wedge \cdots \wedge t(B_{n-1}) = b_{n-1}\}| = 1) \qquad \text{(XER)}$$

where if A is mandatory, then $\pi_A r = A$ must also hold.

2.10. Iris

Lyngbaek and Vianu introduced the Iris data model in [16]. An Iris *schema* is a directed graph combined with a set of constraints. *Literal* and *non-literal types* (entity sets) are represented respectively as labeled square or round nodes in a graph. Relationship sets are called functions and relate the cross product of one group of entity sets to the cross product of another group. *Figure 10* shows how relationship sets are drawn in Iris. Arcs with optional role labels connect a cross-product symbol to entity sets and a directed arrow connects the two cross-product symbols.

The Iris cardinality constraint is called a participation constraint. *Fig. 10* shows two examples. The r: $A[1, \infty]$ cardinality constraint in *Fig. 10* means that every entity in A participates at least once in r, and the constraint r: $A, B[0, 2]$ means that AB-pairs can participate at most twice.

Given an n-ary relationship set r relating entity sets $A_1, \ldots, A_n$, a *participation constraint* takes the form r: $A_1, \ldots, A_p[min, max]$, $p \leq n$, where *min* is 0 or 1 and *max* is any non-negative integer or ∞, and means that each tuple of the cross product $A_1 \times \cdots \times A_p$ must participate in r at least *min* and at most *max* times. When *min* is 0 and *max* is ∞, the cardinality constraint imposes no restriction. The formal definition of an Iris participation

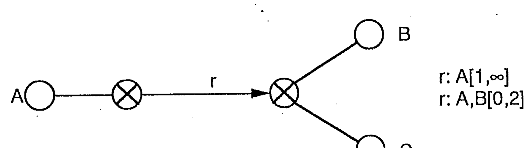

Fig. 10. Iris constraints.

constraint is

$$\forall\langle a_1,\ldots,a_p\rangle \in A_1\times\cdots\times A_p$$
$$(min \leqslant |\{t\in r\colon t(A_1)=a_1\wedge\cdots\wedge t(A_p)=a_p\}| \leqslant max) \qquad \text{(Iris)}$$

where $min \in \{0,1\}$ and $max \in \mathbf{P}\cup\{\infty\}$.

2.11. *Object Modeling Technique*

Object Modeling Technique (OMT) by Rumbaugh et al. is a recent model for object-oriented analysis and design [21]. *Object classes* and *associations* in OMT correspond to entity sets and relationship sets in the ER model. OMT has two methods for defining cardinality constraints: multiplicity for binary relationship sets and candidate keys for n-ary as well as binary relationship sets. *Figure 11* gives examples of multiplicity and candidate-key constraints.

Multiplicity is a generalization of the ER cardinality constraint. Instead of allowing only 1 or many, OMT allows a set of natural numbers to constrain the cardinality. However, OMT only permits multiplicity constraints for binary relationship sets. For example, in *Fig. 11(a)*, entity sets A and B are associated by relationship set r. The multiplicity of A is given as 0, 3–5, which means $\{0,3,4,5\}$, and the multiplicity of B is given as 1+, which means $\{1,2,3,\ldots\}$.

Let r be a binary relationship set associating entity sets A and B and let r have multiplicity s for A. Formally, the OMT multiplicity constraint is defined as

$$\forall b\in B(|\{t\in r\colon t(B)=b\}|\in s)\,. \qquad \text{(OMT-m)}$$

For n-ary relationship sets of degree greater than two, OMT only uses candidate keys to specify cardinality. Candidate keys may also be used for binary relationship sets in addition to multiplicity constraints. The OMT diagram in *Fig. 11(b)* requires AB and AC to be candidate keys for relationship set r. Although not explicitly stated, we assume that an OMT candidate key cannot have zero attributes. The formal definition for candidate keys in OMT is as follows. Let r be an n-ary relationship set and let $\{A_1,\ldots,A_p\}$ be a candidate key for r, $p\leqslant n$, where the A_i's are attributes. Then the following must hold:

$$\forall\langle a_1,\ldots,a_p\rangle \in \pi_{A_1\ldots A_p} r$$
$$(|\{t\in r\colon t(A_1)=a_1\wedge\cdots\wedge t(A_p)=a_p\}|=1)\,. \qquad \text{(OMT-k)}$$

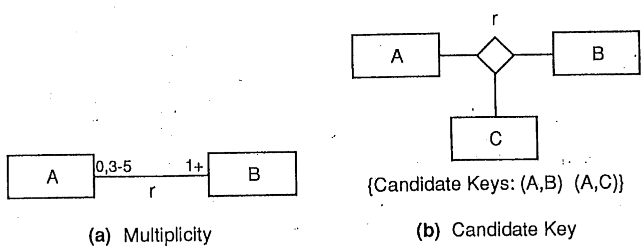

Fig. 11. OMT constraints.

2.12. Object-oriented Systems Analysis

Object-oriented Systems Analysis (OSA) is an object-oriented systems modeling technique that includes a semantic data model [14, 9]. An OSA *object class* corresponds to an entity set in the ER model. OSA defines two relationship-set cardinality constraints: participation constraints, and co-occurrence constraints. *Participation constraints* restrict the number of times an entity may be involved (or participate) in a relationship set. *Co-occurrence constraints* are a generalization of functional dependencies.

In *Fig. 12(a)*, entity sets A and B are associated by relationship set r. A has a participation constraint of $1:1$, which means that each entity in A must participate exactly once in r. B has a participation constraint $1:*$ which means that each entity in B must participate at least once in r and may participate an unlimited number of times. Participation constraint $0:*$ represents no constraint. A participation constraint is specified by a comma-separated list of $min: max$ pairs. It is common for a participation constraint to use a single range of integers, as in *Fig. 12(a)*, but there are also occasions where the more general form is useful.

In *Fig. 12(b)*, entity sets A, B, and C participate in a ternary relationship set r. Each entity in A participates in r between zero and three or between five and seven times, each entity in B participates at least twice, and each entity in C participates zero or more times. However, there is also a co-occurrence constraint

$$AC \text{-}(2:5)\!\!\rightarrow B$$

which means that pairs of entities from A and C that appear in some tuple of r must co-occur (appear together) with at least two and at most five entities from B. If the maximum in *Fig. 12(b)* were $*$ instead of 5, there would be no upper limit on the co-occurrence constraint. A $1:*$ co-ocurrence constraint imposes no restriction. As with OSA participation constraints, the general form of a co-occurrence constraint allows an arbitrary number of comma-separated $min: max$ pairs. Note that min must be at least 1 because any combination of entities for a left side of a co-occurrence constraint either does not appear in a relationship set or appears at least once with some combination of entities for the right side.

We formally define the OSA participation constraint $min_1: max_1, min_2: max_2, \ldots, min_k : max_k$ for entity set A as follows:

$$\begin{aligned} &\forall a \in A((min_1 \leq |\{t \in r: t(A) = a\}| \leq max_1) \\ &\vee \cdots \vee \\ &(min_k \leq |\{t \in r: t(A) = a\}| \leq max_k)) \end{aligned} \qquad \text{(OSA-pc)}$$

where $min_i \in \mathbf{N}$, $max_i \in \mathbf{P} \cup \{*\}$, $1 \leq i \leq k$, and $*$ denotes ∞.

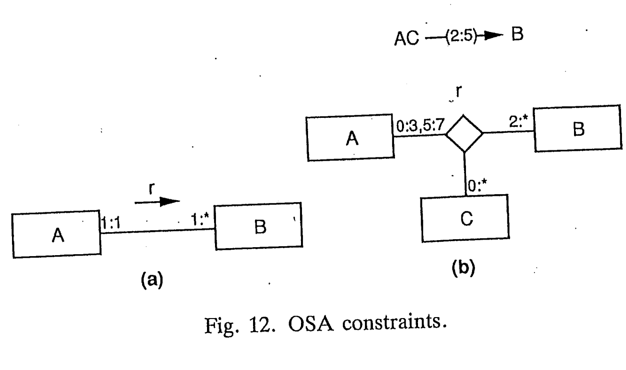

Fig. 12. OSA constraints.

An OSA co-occurrence constraint for relationship set r has the form

$$A_1 \ldots A_p \multimap (min_1: max_1, min_2: max_2, \ldots, min_k: max_k) \rightarrow B_1 \ldots B_q$$

where A_i, $1 \leqslant i \leqslant p$, and B_j, $1 \leqslant j \leqslant q$ are both non-empty sets of entity sets involved in r, and $\{A_1, \ldots, A_p\} \cap \{B_1, \ldots, B_q\} = \varnothing$. The formal definition is

$$\begin{aligned}&\forall \langle a_1, \ldots, a_p \rangle \in \pi_{A_1 \ldots A_p} r \\ &((min \leqslant |\{t \in \pi_{A_1 \ldots A_p B_1 \ldots B_q} r: t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| \leqslant max_1) \\ &\vee \cdots \vee \\ &(min_k \leqslant |\{t \in \pi_{A_1 \ldots A_p B_1 \ldots B_q} r: t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| \leqslant max_k)) \qquad \text{(OSA-co)}\end{aligned}$$

where $min_l \in \mathbf{P}$, $max_l \in (\mathbf{P} \cup \{*\})$, $1 \leqslant l \leqslant k$, and $*$ denotes ∞.

As *Fig. 12(b)* shows, both participation constraints and co-occurrence constraints can be imposed on the same relationship set. Thus, when an n-ary relationship set has a combination of participation and co-occurrence constraints, we have the formula

$$\begin{aligned}&(OSA\text{-}pc_1) \wedge \cdots \wedge (OSA\text{-}pc_n) \wedge \\ &(OSA\text{-}co_1) \wedge \cdots \wedge (OSA\text{-}co_m) \qquad \text{(OSA-c)}\end{aligned}$$

where $(OSA\text{-}pc_i)$, $1 \leqslant i \leqslant n$, is the participation constraint for the ith entity set of r, and $(OSA\text{-}co_j)$, $1 \leqslant j \leqslant m$, $m \geqslant 0$, is a co-occurrence constraint on r.

3. Partial orderings

In this section we present two partial orderings of the cardinality constraints defined in Section 2. The first partial ordering is for cardinality constraints on n-ary relationship sets. The second is for cardinality constraints on binary relationship sets. We also present a universal upper bound for these partial orderings. The precedence relation for the partial orderings is denoted by $\geqslant$. If A and B are semantic models, $A \geqslant B$ means that any cardinality constraint expressible by B has an equivalent representation in A. $A \geqslant B$ can be read as 'A dominates B.' If $A \geqslant B$ and $B \geqslant A$, we say $A \equiv B$.

3.1. N-ary relationship sets

For n-ary relationship sets we consider the following cardinality constraints: ER, SAM, SAM-k, SDM-k, BRM-k, EER-r, ECR, XER, Iris, OMT-k, OSA-pc, OSA-co, and OSA-c. We do not include SBDM, SM, SDM, BRM, or OMT-m because these constraints apply only to binary relationship sets. *Figure 13* gives a diagram of the partial ordering. We now present lemmas and counter examples leading to a theorem that establishes the validity of the partial ordering.

Lemma 1. *SDM-k* $\equiv$ *OMT-k* $\equiv$ *BRM-k*.

Proof. Immediate from the definitions. □

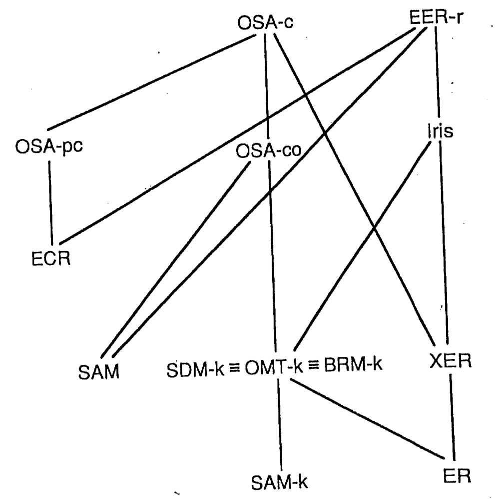

Fig. 13. Partial ordering of cardinality constraints for N-ary relationship sets.

Lemma 2. *OSA-c* $\geqslant$ *OSA-pc and OSA-c* $\geqslant$ *OSA-co.*

Proof. Immediate from the definitions. □

Lemma 3. *OSA-c* $\geqslant$ *XER.*

Proof. Given an arbitrary XER cardinality constraint, we show how to construct an equivalent OSA-c cardinality constraint. Consider an XER cardinality constraint on a relationship set r. Let E be the n entity sets for r, and let $E' \subseteq E$ be the entity sets of E attached to unshaded corners. For each $A \in E'$, letting $\{B_1, B_2, \ldots, B_{n-1}\} = E - A$, we write the OSA-co constraint $B_1B_2 \ldots B_{n-1} \multimap(1:1)\!\!\rightarrow A$. Noting that $1 \leqslant x \leqslant 1$ implies $x = 1$, and that $\pi_{B_1B_2 \ldots B_{n-1}A}r = r$, this co-occurrence constraint reduces (OSA-co) to

$$\forall \langle b_1, \ldots, b_{n-1} \rangle \in \pi_{B_1 \ldots B_{n-1}} r$$
$$(|\{t \in r : t(B_1) = b_1 \wedge \cdots \wedge t(B_{n-1}) = b_{n-1}\}| = 1)$$

which is (XER). However, we must also account for the possibility of A being mandatory. If A is optional, we write the OSA-pc constraint $0{:}*$ for A, otherwise write the OSA-pc constraint $1{:}*$ for A. By (OSA-pc) the participation constraint $1{:}*$ for A is

$$\forall a \in A(1 \leqslant |\{t \in r : t(A) = a\}|)$$

which implies that $\pi_A r = A$ given our assumption of referential integrity. □

Lemma 4. *EER-r* $\geqslant$ *Iris.*

Proof. Consider an Iris cardinality constraint, r: $A_1, \ldots, A_p[min, max]$, on a relationship set r. We construct an equivalent EER-r constraint as follows. Let E be the set of entity sets participating in r, let $X = \{A_1, \ldots, A_p\}$ be a subset of E, and let $Y = \{B_1, \ldots, B_q\}$ be

$E - X$. We now write the EER-r constraints $Cmin[r(X/Y] = 1$ and $Cmax[r(X/Y)] = max$. Observing that $\pi_{A_1 \ldots A_p B_1 \ldots B_q} r = r$, and substituting in (EER-r), we obtain

$$\forall \langle a_1, \ldots, a_p \rangle \in \pi_{A_1 \ldots A_p} r$$
$$(1 \leq |\{t \in r\colon t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| \leq max) \quad (1)$$

We now consider the two cases for the Iris *min*. If $min = 1$, then we specify that *Cmin* is not optional. By the definition of EER-r, this means that $A_1 \times \cdots \times A_p = \pi_{A_1 \ldots A_p} r$. Substitution of the cross product for the projection in (1) yields (Iris), where the Iris $min = 1$. Now suppose that the Iris $min = 0$. In this case we specify *Cmin* to be optional; thus $\pi_{A_1 \ldots A_p} r$ need not be the same as $A_1 \times \cdots \times A_p$. Thus, in addition to (1), we have

$$\forall \langle a_1, \ldots, a_p \rangle \in (A_1 \times \cdots \times A_p - \pi_{A_1 \ldots A_p} r)$$
$$(|\{t \in r\colon t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| = 0)\,. \quad (2)$$

Consider a tuple $t' \in A_1 \times \cdots \times A_p$. Either t' is in $\pi_{A_1 \ldots A_p} r$, or it is in $A_1 \times \cdots \times A_p - \pi_{A_1 \ldots A_p} r$. Therefore, the cardinality of the set of tuples in r that have the same A_1- . . . A_p-values as t', as restricted by (1) and (2), is between the Iris *min* (=0 in this case) and the Iris *max*. □

Lemma 5. *Iris* $\geq$ *XER.*

Proof. Consider an XER cardinality constraint on a relationship set r. Let E be the set of entity sets for r, and let $E' \subseteq E$ be the entity sets of E attached to unshaded corners. We construct an equivalent Iris constraint as follows. For each $A \in E'$, let $E - A = B_1 \ldots B_{n-1}$. Now write the constraint $r\colon B_1, \ldots, B_{n-1}[0, 1]$, and, if A is mandatory, write the constraint $A[1, \infty]$. Substituting $r\colon B_1, \ldots, B_{n-1}[0, 1]$ in (Iris), we obtain

$$\forall \langle b_1, \ldots, b_{n-1} \rangle \in B_1 \times \cdots \times B_{n-1}(|\{t \in r\colon t(B_1) = b_1 \wedge \cdots \wedge t(B_p) = b_{n-1}\}| \leq 1)\,.$$

Since $\pi_{A_1 \ldots A_p} r \subseteq A_1 \times \cdots \times A_p$, this implies

$$\forall \langle b_1, \ldots, b_{n-1} \rangle \in \pi_{B_1 \ldots B_{n-1}} r(|\{t \in r\colon t(B_1) = b_1 \wedge \cdots \wedge t(B_{n-1}) = b_{n-1}\}| = 1)$$

which is (XER). When A is mandatory, by substituting $r\colon A[1, \infty]$ in (Iris), we also obtain

$$\forall a \in A(1 \leq |\{t \in r\colon t(A) = a\}| \leq \infty)$$

which, because of referential integrity, implies that $\pi_A r = A$. □

Lemma 6. *Iris* $\geq$ *SDM-k.*

Proof. Let $K = \{A_1, \ldots, A_p\}$ be an SDM-k candidate-key constraint on relationship set r. Consider the Iris constraint $r\colon A_1 \ldots A_p[0,1]$, which by substitution in (Iris) is

$$\forall \langle a_1, \ldots, a_p \rangle \in A_1 \times \cdots \times A_p(|\{t \in r\colon t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| \leq 1)\,.$$

Since $\pi_{A_1 \ldots A_p} r \subseteq A_1 \times \cdots \times A_p$, this implies

$$\forall\langle a_1, \ldots, a_p\rangle \in \pi_{A_1 \ldots A_p} r(|\{t \in r\colon t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| = 1)$$

which is (SDM-k). □

Lemma 7. *OSA-co ⩾ SDM-k.*

Proof. Let r be a relationship set on scheme $R = \{A_1, \ldots, A_n\}$, and let $K = \{A_1, \ldots, A_p\}$, $p \leqslant n$, be an SDM-k candidate-key constraint. If $K = R$, then K represents no constraint since r is a set. Therefore, we assume without loss of generality that K is a proper subset of R. Now consider the OSA-co constraint $K \text{-}(1:1)\!\!\rightarrow R - K$. Noting that $1 \leqslant x \leqslant 1$ implies $x = 1$, by substitution in (OSA-co) this is

$$\forall\langle a_1, \ldots, a_p\rangle \in \pi_{A_1 \ldots A_p} r(|\{t \in \pi_R r\colon t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| = 1)$$

which is (SDM-k). □

Lemma 8. *OSA-pc ⩾ ECR.*

Proof. Immediate from the definitions. (ECR) is a special case of (OSA-pc), where there is only one *min*: *max* range and *max* is constrained to be greater than or equal to *min*. □

Lemma 9. *EER-r ⩾ ECR.*

Proof. Let r be an n-ary relationship set with schema $R = \{A_1, \ldots, A_n\}$. Given an ECR constraint (min_i, max_i) on A_i, $1 \leqslant i \leqslant n$, we construct an equivalent EER-r constraint as follows. If $min_i = 0$ then let $Cmin[r(A_i/R - A_i)]$ be optional, else let $Cmin[r(A_i/R - A_i)] = min_i$, where *Cmin* is not optional. Now let $Cmax[r(A_i/R - A_i)] = max_i$. Substituting this instance of *Cmin* and *Cmax* into (EER-r), and noting that $\pi_{A_i(R-A_i)} r = r$, we obtain

$$\forall\langle a_i\rangle \in \pi_{A_i} r(min_i \leqslant |\{t \in r\colon t(A_i) = a_i\}| \leqslant max_i) \tag{3}$$

for every i, $1 \leqslant i \leqslant n$. If *Cmin* is not optional, then $\pi_{A_i} r = A_i$, and since $\langle a_i\rangle = a_i$, substitution yields

$$\forall a_i \in A_i(min_i \leqslant |\{t \in r\colon t(A_i) = a_i\}| \leqslant max_i)$$

which is (ECR). If *Cmin* is optional, then we know $min_i = 0$; thus $\pi_{A_i} r$ need not be the same as A_i. Thus, in addition to (3), we have

$$\forall\langle a_i\rangle \in (A_i - \pi_{A_i} r)$$
$$(|\{t \in r\colon t(A_i) = a_i\}| = 0)\,. \tag{4}$$

Consider a tuple $t' \in A_i$. Either t' is in $\pi_{A_i} r$, or it is in $A_i - \pi_{A_i} r$. Therefore, the cardinality of the set of tuples in r that have the same A_i-values as t', as restricted by (3) and (4), is between the ECR min_i (=0 in this case) and the ECR max_i. □

Lemma 10. *EER-r ⩾ SAM.*

Proof. An $m-n$ SAM constraint imposes no constraint, and so we need not examine the $m-n$ case further. Also, since a $1-1$ SAM constraint from A to B is the same as a $1-m$ SAM constraint from A to B and a $1-m$ SAM constraint from B to A, we need only consider the $1-m$ case. For each $1-m$ SAM constraint from A to B in an n-ary relationship set r, where A and B are distinct entity sets involved in r, we write the EER-r constraint $Cmax[r(B/A)]=1$. This yields

$$\forall\langle b\rangle \in \pi_B r(|\{t \in \pi_{AB} r\colon t(B)=b\}|=1)$$

which is (SAM). □

Lemma 11. *OSA-co ⩾ SAM.*

Proof. Similar to proof of Lemma 10, using B –(1: 1)→ A for $Cmax[r(B/A)]=1$. □

Lemma 12. *SDM-k ⩾ SAM-k.*

Proof. Immediate from the definitions. The difference is that SAM-k constraints are restricted to one per relationship set, whereas the number of SDM-k constraints per relationship set is unrestricted. □

Lemma 13. *SDM-k ⩾ ER.*

Proof. Consider an ER constraint on an n-ary relationship set r. Let E be the set of entity sets for r, and let $E' \subseteq E$ be the set of entity sets in E marked by the mapping symbol 1. For each $A \in E'$, we write the SDM-k candidate key constraint $E-A$. For each such SDM-k constraint, letting $E-A=\{B_1, \ldots, B_{n-1}\}$, we have

$$\forall\langle b_1, \ldots, b_{n-1}\rangle \in \pi_{B_1 \ldots B_{n-1}} r$$
$$(|\{t \in r\colon t(B_1)=b_1 \wedge \cdots \wedge t(B_{n-1})=b_{n-1}\}|=1)$$

which is (ER). □

Lemma 14. *XER ⩾ ER.*

Proof. Immediate from the definitions. (XER) is (ER) with the addition of the ability to specify mandatory connections. □

We now have all the lemmas necessary to establish the domination relationships of our n-ary partial-ordering result. However, to show that there are no more domination relationships in the partial ordering, we also need the counter examples in *Table 1*. Observe that some counter examples that might be expected are not needed. For example, ER ⋡ OSA-pc, since OSA-pc ⩾ ECR and ER ⋡ ECR together imply ER ⋡ OSA-pc; for if not, then ER ⩾ ECR. In general, we need counter examples only for the least lower bounds of non-dominated models. For example, for ER we need counter examples for only SAM, SAM-k, ECR, and XER.

Theorem 1. *The set of models* {*ER, SAM, SAM-k, SDM-k, BRM-k, EER-r, ECR, XER,* *Iris, OMT-k, OSA-pc, OSA-co, OSA-c} together with the precedence relation ⩾ satisfies the partial ordering in Fig. 13.*

Proof. Lemmas 1–14 and Counter Examples 1–30. □

3.2. Binary relationship sets

For binary relationship sets we consider the following cardinality constraints: SBDM, ER, SM, SAM, SAM-k, SDM, SDM-k, BRM, BRM-k, EER-r, ECR, XER, Iris, OMT-m, OMT-k, OSA-pc, OSA-co, and OSA-c. *Figure 14* gives a diagram of the partial ordering for

Table 1
Counter examples for N-ary relationship-set results

#	≱	Counter Example*	Notes
	EER-r		
1	≱ OSA-pc	1:1,3:* OSA-pc on A	EER-r cannot restrict participation to a disjoint range of integers.
2	≱ OSA-co	A–(1:1,3:*)>B	#1
	OSA		
3	≱ Iris	r: A, B [1,1]	OSA cannot ensure that r is the cross product of A and B.
	Iris		
4	≱ ECR	(2,∞) ECR on A	Iris cannot ensure that entities in A participate more than once.
5	≱ SAM	Let r be ternary on A, B, and C; 1-m from A to B.	Iris cannot enforce a functional dependency that involves only some of the entity sets of r.
	OSA-pc		
6	≱ SAM	#5	OSA-pc cannot constrain participation of co-occurrences, only participation of single-entity occurrences.
7	≱ SAM-k	Let r be ternary on A, B, and C with key AB.	#6
8	≱ ER	Let r be ternary on A, B, and C with 1 on C.	#6
	ECR		
9	≱ OSA-pc	#1	ECR cannot restrict participation to a disjoint range of integers.
10	≱ SAM	#5	ECR cannot constrain participation of co-occurrences, only participation of single-entity occurrences.
11	≱ SAM-k	#7	#10
12	≱ ER	#8	#10
	OSA-co		
13	≱ ECR	(1,∞) ECR on A	OSA-co cannot ensure that entities in A participate in r.
14	≱ XER	A mandatory	#13
	SDM-k≡OMT-k≡BRM-k		
15	≱ ECR	#4	SDM-k cannot ensure that entities in A participate more than once.
16	≱ SAM	#5	SDM-k cannot enforce a functional dependency that involves only some of the entity sets of r.
17	≱ XER	#14	SDM-k cannot ensure that entities in A participate in r.

*Counter examples assume a binary relationship set r involving entity sets A and B, unless otherwise specified.

Table 1 (cont.)
Counter examples for *N*-ary relationship-set results

#	$\not\geq$	Counter Example*	Notes
	XER		
18	$\not\geq$ ECR	#4	XER cannot ensure that entities in *A* participate more than once.
19	$\not\geq$ SAM	#5	XER cannot enforce a functional dependency that involves only some of the entity sets of *r*.
20	$\not\geq$ SAM-k	Let *r* be ternary on *A*, *B*, and *C* with key *A*.	For *n*-ary relationship sets, XER keys involve *n*–1 entity sets.
	SAM		
21	$\not\geq$ ECR	#4	SAM cannot ensure that entities in *A* participate more than once.
22	$\not\geq$ SAM-k	#7	SAM cannot enforce composite candidate-key constraints.
23	$\not\geq$ ER	#8	#22
	ER		
24	$\not\geq$ SAM	#5	ER cannot ensure participation of entities in *A* with respect to *B*, only with respect to all of *r*.
25	$\not\geq$ SAM-k	#20	For *n*-ary relationship sets, ER keys involve *n*–1 entity sets.
26	$\not\geq$ ECR	#4	ER cannot ensure that entities in *A* participate more than once.
27	$\not\geq$ XER	#14	#26
	SAM-k		
28	$\not\geq$ ER	1's on *A* and *B*	SAM-k can only specify one key per relationship set.
29	$\not\geq$ SAM	1-1 from *A* to *B*	This implies that *A* and *B* are both keys. But SAM-k can only specify one key per relationship set.
30	$\not\geq$ ECR	#4	SAM-k cannot ensure that entities in *A* participate more than once.

*Counter examples assume a binary relationship set *r* involving entity sets *A* and *B*, unless otherwise specified.

cardinality constraints restricted to binary relationship sets. We now present lemmas and counter examples leading to a theorem that establishes the validity of this partial ordering.

Lemma 15. *OSA-pc* $\geqslant$ *OSA-co for binary relationship sets.*

Proof. Given an OSA-co constraint for a binary relationship set, we show how to construct an equivalent OSA-pc constraint. Let r be a binary relationship set involving entity sets A and B. An OSA-co constraint has the form $A \dashv\!(min_1: max_1, \ldots, min_k: max_k)\!\!\rightarrowtail B$. Observing that for a binary relationship set, $\pi_{AB}r = r$, (OSA-co) reduces to

$$\forall\langle a\rangle \in \pi_A r((min_1 \leqslant |\{t \in r: t(A) = a\}| \leqslant max_1) \vee \cdots \vee (min_k \leqslant |\{t \in r: t(A) = a\}| \leqslant max_k)) .$$

Now, for the participation constraint: $0{:}0$, $min_1: max_1, \ldots, min_k: max_k$, (OSA-pc) is

$$\forall a \in A((|\{t \in r: t(A) = a\}| = 0) \vee (min_1 \leqslant |\{t \in r: t(A) = a\}| \leqslant max_1) \vee \cdots \vee (min_k \leqslant |\{t \in r: t(A) = a\}| \leqslant max_k)) .$$

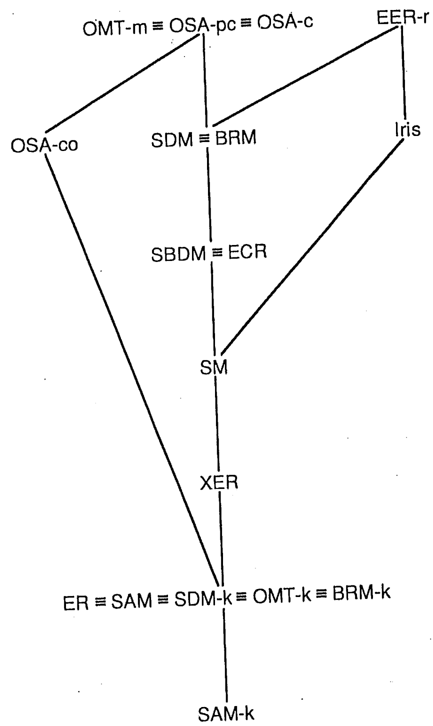

Fig. 14. Partial ordering of cardinality constraints for binary relationship sets.

Since we are assuming referential integrity, and since whenever $\langle a \rangle \in \pi_A r$, $|\{t \in r : t(A) = a\}| > 0$, the second formula implies the first, and thus, OSA-pc $\geqslant$ OSA-co. A similar construction holds for an OSA-co constraint $B \multimap (min_1 : max_1, \ldots, min_k : max_k) \mapsto A$. □

Lemma 16. *OMT-m ≡ OSA-pc ≡ OSA-c for binary relationships sets.*

Proof. We show first that OMT-m ≡ OSA-pc, and then that OSA-pc ≡ OSA-c. Let r be a binary relationship set associating entity sets A and B. Let r have OMT multiplicity s for B, then by (OMT-m),

$$\forall a \in A(|\{t \in r : t(A) = a\}| \in s) .$$

But this is equivalent to

$$\forall a \in A((min_1 \leqslant |\{t \in r : t(A) = a\}| \leqslant max_1) \vee \cdots \vee$$
$$(min_k \leqslant |\{t \in r : t(A) = a\}| \leqslant max_k))$$

where $s = \{min_1 : max_1, \ldots, min_k : max_k\}$, which is (OSA-pc) for $min_1 : max_1, \ldots, min_k : max_k$ for A. Thus, OMT-m ≡ OSA-pc.

By Lemma 15 OSA-pc $\geqslant$ OSA-co for binary relationship sets. Thus, we can replace each OSA-co constraint in (OSA-c) with an equivalent OSA-pc constraint, as shown in Lemma 15. The result is a conjunction of only OSA-pc constraints, and hence, OSA-pc ≡ OSA-c. □

Lemma 17. *$SDM \equiv BRM$ for binary relationship sets.*

Proof. Immediate from the definitions, noting that $max = \infty$ in (SDM) is the same as omitting *max* for (BRM). □

Lemma 18. *$SBDM \equiv ECR$ for binary relationship sets.*

Proof. Immediate from the definitions. □

Lemma 19. *$ER \equiv SAM \equiv SDM\text{-}k \equiv OMT\text{-}k \equiv BRM\text{-}k$ for binary relationship sets.*

Proof. Let r be a binary relationship set involving entity sets A and B. The table below shows correspondences among the four possible OMT-k candidate keys, the four possible A to B ER mappings, and the four possible A to B SAM mappings. The table also shows the OMT-k constraint, the ER constraint, and the SAM constraint, which by (OMT-k), (ER), and (SAM) are all identical. □

OMT-k	ER	SAM	Constraint
$\{(AB)\}$	$M:N$	$m-n$	None
$\{(A)\}$	$M:1$	$m-1$	$\forall\langle a\rangle \in \pi_A r(\lvert\{t \in r: t(A)=a\}\rvert = 1)$
$\{(B)\}$	$1:M$	$1-m$	$\forall\langle b\rangle \in \pi_B r(\lvert\{t \in r: t(B)=b\}\rvert = 1)$
$\{(A),(B)\}$	$1:1$	$1-1$	$\forall\langle a\rangle \in \pi_A r(\lvert\{t \in r: t(A)=a\}\rvert = 1) \wedge$ $\forall\langle b\rangle \in \pi_B r(\lvert\{t \in r: t(B)=b\}\rvert = 1)$

Lemma 20. *$OSA\text{-}pc \geqslant SDM$ for binary relationship sets.*

Proof. Let r be a binary relationship set involving entity sets A and B. Since R single-valued is a special case of R multivalued, we assume without loss of generality that B has a multivalued attribute r that maps entities of B to entities of A with cardinality constraints *min* and *max*. If *min* and *max* are positive integers, then we write the OSA-pc constraint 0:0, min:max for A; if the SDM *max* is ∞, then we write the OSA-pc *max* as $*$. This yields

$$\forall a \in A((\lvert\{t \in r: t(A) = a\}\rvert = 0) \vee (min \leqslant \lvert\{t \in r: t(A) = a\}\rvert \leqslant max))$$

which as in Lemma 15 implies

$$\forall\langle a\rangle \in \pi_A r(min \leqslant \lvert\{t \in r: t(A) = a\}\rvert \leqslant max)$$

which is (SDM). We must also consider whether the SDM constraint exhausts B. If not, the formula stands as specified. If so, then we must have $\pi_B r = B$, which can be written as an additional OSA-pc constraint with a minimum participation of at least 1 on B. Since one SDM constraint may be equivalent to two OSA-pc constraints on separate entity sets, we must consider the interaction between multiple constraints. First suppose that the inverse attribute of r does not exist. Then the OSA-pc constraint on B would be $1:*$. Now suppose that the inverse attribute of r were multivalued with size between min_b and max_b. Then the OSA-pc constraint on B would be $min_b : max_b$. Since $min_b \in \mathbf{P}$, this OSA-pc constraint will ensure that $\pi_B r = B$. If the inverse attribute were required to exhaust A, then we must strengthen the OSA-pc constraint on A to $min: max$. This yields

$$\forall a \in A(min \leqslant |\{t \in r: t(A) = a\}| \leqslant max)$$

and since $a \in A$ implies $\langle a \rangle \in \pi_A r$, this formula implies (SDM). □

Lemma 21. *EER-r ⩾ SDM for binary relationship sets.*

Proof. As in the proof of Lemma 20, we assume a binary relationship set r involving entity sets A and B, with an SDM constraint that B be multivalued with size between min and max. We write the equivalent EER-r constraint as $Cmin[r(A/B)] = min$ and $Cmax[r(A/B)] = max$, with $Cmin$ optional. Noting that $\pi_{AB} r = r$, then by (EER-r),

$$\forall \langle a \rangle \in \pi_A r(min \leqslant |\{t \in r: t(A) = a\}| \leqslant max)$$

which is (SDM). If the SDM constraint must exhaust B, then we must also have $\pi_B r = B$. As in the proof of Lemma 20, there are several cases for interaction of SDM constraints. If the inverse attribute for r does not exist, then we specify $Cmin[r(B/A)] = 0$ with $Cmin$ not optional, which forces $\pi_B r = B$ while imposing no additional constraint on r. If the inverse attribute for r is multivalued with size between min_b and max_b, then we would impose the additional EER-r constraint $Cmin[r(B/A)] = min_b$. Furthermore, if the inverse attribute exhausts A, then we would require that $Cmin[r(A/B)]$ not be optional, thus forcing $\pi_B r = B$. The case is similar for an SDM constraint that A be multivalued. □

Lemma 22. *SDM ⩾ ECR for binary relationship sets.*

Proof. Let r be a binary relationship set involving entity sets A and B. Given an ECR constraint (min, max) on A, we construct an equivalent SDM constraint as follows. We must consider two cases: $min = 0$ and $min \neq 0$. First, suppose $min = 0$, which means that entities in A need not participate in r. Let A be multivalued with size between min and max. By (SDM), this yields

$$\forall \langle a \rangle \in \pi_A r(min \leqslant |\{t \in r: t(A) = a\}| \leqslant max), \tag{5}$$

but since $min = 0$ and any $\langle a \rangle \not\in \pi_A r$ participates zero times in r, then (5) implies

$$\forall a \in A(min \leqslant |\{t \in r: t(A) = a\}| \leqslant max) \tag{6}$$

which is (ECR). Now suppose $min \neq 0$; then the ECR constraint requires that each entity in A participate in r, or in other words, $\pi_A r = A$. Then by substitution, (5) yields (6) again, which is (ECR). □

Lemma 23. *Iris ⩾ SM for binary relationship sets.*

Proof. Let r be a binary relationship set involving entity sets A and B. Given an SM constraint $m: n$, where m applies to A and n to B, we construct an equivalent Iris constraint as follows. Consider Iris constraints $r: A[0, m]$ and $r: B[0, n]$; by (Iris), this yields

$$\begin{aligned} &\forall \langle a \rangle \in A(|\{t \in r: t(A) = a\}| \leqslant m) \wedge \\ &\forall \langle b \rangle \in B(|\{t \in r: t(B) = b\}| \leqslant n) \end{aligned} \tag{7}$$

We now must consider several cases. First, suppose the SM dependency is partial with respect to both A and B. Then $\pi_A r$ need not be the same as A. In this case we observe that (7) is equivalent to (SM), noting that any $\langle a \rangle \not\in \pi_A r$ will appear in no tuple of r (which is less than m), and any $\langle a \rangle \in \pi_A r$ can appear at most m times. Similarly, $\pi_B r$ need not be the same as B. Second, suppose that the dependency is partial only with respect to B. Then (SM) requires that $\pi_A r = A$. We strengthen the Iris constraint from $r\colon A[0, m]$ to $r\colon A[1, m]$, which ensures that each entity in A will participate in r, or that $\pi_A r = A$ as desired. The case is similar if the dependency is partial only with respect to A. Third, suppose the SM dependency is total. Then we strengthen both Iris constraints to have a lower bound of 1 as in the previous case, ensuring that $\pi_A r = A$ and $\pi_B r = B$. □

Lemma 24. *ECR ⩾ SM for binary relationship sets.*

Proof. Assume a binary relationship set r and SM constraint $m\colon n$ as in the proof of Lemma 23. Consider the ECR constraints $(0, m)$ on A and $(0, n)$ on B, which by (ECR) yield

$$\forall a \in A(0 \leqslant |\{t \in r\colon t(A) = a\}| \leqslant m) \wedge$$
$$\forall b \in B(0 \leqslant |\{t \in r\colon t(B) = b\}| \leqslant n)\,.$$

Since the cardinality of any set is nonnegative, this implies

$$\forall a \in A(|\{t \in r\colon t(A) = a\}| \leqslant m) \wedge$$
$$\forall b \in B(|\{t \in r\colon t(B) = b\}| \leqslant n)$$

which is equivalent to (7). Following reasoning similar to that of the proof of Lemma 23, these ECR constraints are equivalent to the given SM constraint when there is no SM dependency (i.e., the dependency is partial with respect to both A an B). If the SM dependency is partial only with respect to A, then we satisfy the requirement that $\pi_B r = B$ by strengthening the ECR constraint on B to $(1, n)$. Similarly, if the dependency is partial only with respect to B, then we strengthen the ECR constraint on A to $(1, m)$. Finally, if the SM dependency is total, we strengthen both ECR constraints. □

Lemma 25. *SM ⩾ XER for binary relationship sets.*

Proof. Let r be a binary relationship set involving entity sets A and B. There are several possible XER constraints on r. First, suppose both corners are shaded; this represents no constraint. Second, suppose only one corner is shaded, say the corner to which B is attached. Then we create an SM cardinality constraint of 1 with A and ∞ with B. By (SM), this is

$$\forall \langle a \rangle \in \pi_A r(|\{t \in r\colon t(A) = a\}| \leqslant 1) \wedge$$
$$\forall \langle b \rangle \in \pi_B r(|\{t \in r\colon t(B) = b\}| \leqslant \infty)\,.$$

By definition, an entity in $\pi_A r$ appears in at least one tuple of r; also, the second conjunct represents no constraint. Thus, the above formula reduces to

$$\forall \langle a \rangle \in \pi_A r(|\{t \in r\colon t(A) = a\}| = 1)$$

which is (XER) for $n = 2$. The case is similar if only the corner attached to A is shaded. Third, if both corners are unshaded, then we create an SM cardinality constraint of 1 with both A and B. This yields

$$\forall\langle a\rangle \in \pi_A r(|\{t \in r : t(A) = a\}| = 1) \wedge$$
$$\forall\langle b\rangle \in \pi_B r(|\{t \in r : t(B) = b\}| = 1)$$

which is two instances of (XER) for $n = 2$. Now we must consider the effect of optional versus mandatory connections. The corresponding concept in SM is partial versus total

Table 2
Counter examples for binary relationship-set results

#	≢	Counter Example*	Notes
	EER-r		
31	≢ OSA-co	#2	#1
	Iris		
32	≢ OSA-co	A–(2:*)>B	Iris cannot ensure that entities in A that participate do so more than once.
33	≢ SBDM≡ECR	#4	#4
	OMT-m≡OSA-pc		
34	≢ Iris	#3	OMT-m≡OSA-pc cannot ensure that r is the cross product of A and B.
	OSA-co		
35	≢ XER	#14	#13
	SBDM≡ECR		
36	≢ Iris	#3	SDBM≡ECR cannot ensure that r is the cross product of A and B.
37	≢ OSA-co	#32	SBDM≡ECR cannot both allow non-participation of entities in A and ensure at least 2 occurrences.
38	≢ SDM≡BRM	Let B be multivalued with size between 2 and 4.	#37
	SDM≡BRM		
39	≢ Iris	#3	SDM≡BRM cannot ensure that r is the cross product of A and B.
40	≢ OSA-co	#32	SDM≡BRM cannot both allow non-participation of entities in A and ensure at least 2 occurrences.
	SM		
41	≢ OSA-co	#32	SM cannot ensure that entities in A that participate do so more than once.
42	≢ SBDM≡ECR	#4	#41
43	≢ Iris	#3	SM cannot ensure that r is the cross product of A and B.
	XER		
44	≢ OSA-co	#32	XER cannot ensure that entities in A that participate do so more than once.
45	≢ SM	Total dependency	XER cannot ensure that entities in A participate in r.
	ER≡SAM≡SDM-k ≡OMT-k≡BRM-k		
46	≢ OSA-co	#32	ER≡... cannot ensure that entities in A that participate do so more than once.
47	≢ XER	#14	#26
	SAM-k		
48	≢ ER≡...	#28	#28

*Counter examples assume a binary relationship set r involving entity sets A and B.

dependencies. In XER, a mandatory connection on A requires that $\pi_A r = A$; in SM, if the relationship set r is partial only with respect to B, then $\pi_A r = A$. Thus, if only the connection to A is optional, then the SM dependency is partial only with respect to A, and similarly if only B is optional. If both are optional, then the SM dependency is partial with respect to both A and B. If both are mandatory, then the SM dependency is total. □

We now have all the lemmas necessary to establish the domination relationships of our binary partial-ordering result. However, to show that there are no more domination relationships, we also need the counter examples in *Table 2*.

Theorem 2. *The set of constraints {SBDM, ER, SM, SAM, SAM-k, SDM, SDM-k, BRM, BRM-k, EER-r,ECR, XER, Iris, OMT-m, OMT-k, OSA-pc, OSA-co, OSA-c} together with the precedence relation ⩾ satisfies the partial ordering in Fig. 14.*

Proof. Lemmas 1, 4, 11, 12, 14–25 and Counter Examples 31–48. □

4. Entity-set cardinality constraints

Until now we have only considered relationship-set cardinality constraints. However, two of the models studied, EER and OSA, also have entity-set cardinality constraints. In this section, we give formal definitions for the entity-set cardinality constraints of EER and OSA, and we show how these constraints, in combination with the relationship-set constraints defined earlier, modify the partial orderings of Section 3.

4.1. Enriched Entity-Relationship Model

In the EER model, an *absolute cardinality constraint* specifies a minimum and maximum number of instances for a given entity or relationship set. For example, suppose we have 500 parking spaces available; we can specify that the entity set representing parking spaces has a maximum cardinality, *Cmax*, of 500 (we can also assign a minimum cardinality, *Cmin*). As another example, suppose that we have an unspecified number of parking spaces, and an unspecified number of vehicles that may be parked; however, there is a limit of 20 reserved parking spaces. We can model this with a relationship set *Is Assigned To* between entity sets *Vehicle* and *Parking Space*, together with an absolute cardinality constraint $Cmax = 20$ on the relationship set.

Let S be an entity set or a relationship set. Given $Cmin[S] = min$ and $Cmax[S] = max$, the following must hold:

$$min \leqslant |S| \leqslant max \qquad \text{(EER-a)}$$

where min, $max \in \mathbf{P}$. Note that *Cmin* or *Cmax* may be omitted, in which case the corresponding inequality of (EER-a) is ignored.

Since EER absolute and relative cardinality constraints can appear together for an n-ary relationship set r, we have the formula

$$(EER\text{-}r_1) \wedge \cdots \wedge (EER\text{-}r_p) \wedge (EER\text{-}a_1) \wedge \cdots \wedge (EER\text{-}a_q) \qquad \text{(EER)}$$

where $(EER\text{-}r_i)$, $1 \leqslant i \leqslant p$, is a relative cardinality constraint for r, and $(EER\text{-}a_j)$, $1 \leqslant j \leqslant q$, is an absolute cardinality constraint on r or an entity set involved in r.

4.2. Object-oriented systems analysis

In OSA, an *object-class cardinality constraint* restricts the number of entities that may belong to a particular entity set. In *Fig. 15*, A has an object-class cardinality constraint of 2:10, which means that the number of entities in A must be between 2 and 10. Another feature of the OSA semantic model is that each relationship set can be viewed as an entity set; this is done using a relational object class. A *relational object class* is an entity set whose members are in a one-to-one correspondence with the elements of the corresponding relationship set. Thus, through the relational object class, OSA's object-class cardinality constraint can be applied to relationship sets.

Formally, an OSA object-class cardinality constraint for entity set A is of the form $min_1 : max_1, min_2 : max_2, \ldots, min_k : max_k$, and is defined as follows:

$$\begin{aligned} & min_1 \leq |A| \leq max_1 \\ & \vee \cdots \vee \\ & min_k \leq |A| \leq max_k \end{aligned} \qquad \text{(OSA-o)}$$

where $min_i \in \mathbf{N}$, $max_i \in (\mathbf{N} \cup \{*\})$, $1 \leq i \leq k$, and $*$ denotes ∞.

In addition to object-class cardinality constraints, OSA also supports one other feature involving object-class cardinalities. The domains of the $min_i : max_i$ ranges in the definitions of (OSA-pc), (OSA-o), and (OSA-co) are augmented to include entity-set cardinalities. For example, the participation constraint on B in *Fig. 15* is $|A|$ (shorthand for $|A|$:$|A|$), indicating that each entity in B must relate through r to $|A|$ entities in A, in other words, to all entities in A. Similarly, because of the participation constraint on A, each entity in A must relate to all entities in B. In this example, because of the participation constraints on A and B, r must equal the full cross product of A and B. Entity-set cardinalities used in OSA cardinality constraints need not be restricted to entity sets participating in the same relationship set. For example, the object-class cardinality constraint on B in *Fig. 15* indicates that the cardinality of entity set B must be the same as the cardinality of entity set C.

Since OSA relationship-set and entity-set cardinality constraints may appear in combination for an n-ary relationship set r, we have the formula

$$\begin{aligned} & (OSA\text{-}pc_1) \wedge \cdots \wedge (OSA\text{-}pc_n) \wedge \\ & (OSA\text{-}co_1) \wedge \cdots \wedge (OSA\text{-}co_m) \wedge \\ & (OSA\text{-}o_1) \wedge \cdots \wedge (OSA\text{-}o_n) \end{aligned} \qquad \text{(OSA)}$$

where $(OSA\text{-}pc_i)$ and $(OSA\text{-}o_i)$, $1 \leq i \leq n$, are participation and object-class cardinality constraints respectively for the ith entity set of r, and $(OSA\text{-}co_j)$, $1 \leq j \leq m$, is a co-occurrence constraint for r, and where $min_i \in \mathbf{N} \cup \{|A|$: *A is an entity set*$\}$, $max_i \in \mathbf{N} \cup$

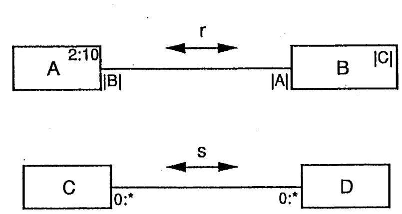

Fig. 15. OSA entity-set cardinality constraints.

$\{*\} \cup \{|A|: A \textit{ is an entity set}\}$, $min_j \in \mathbf{P} \cup \{|A|: A \textit{ is an entity set}\}$, and $max_j \in \mathbf{P} \cup \{*\} \cup \{|A|: A \textit{ is an entity set}\}$.

As an aside, the shorthand ($|A|$ for $|A|{:}|A|$) holds universally in OSA. Thus, for example, an OSA user may write 1 for 1:1 or 2 for 2:2 in any participation constraint, co-occurrence constraint, or object-class cardinality constraint.

4.3. *Modified partial-ordering results*

We now explore how the addition of entity-set cardinality constraints affects the partial orderings of Section 3. We do this by adding EER and OSA to the partial orderings. Our next lemmas show that EER dominates EER-r but not OSA-c, and that OSA is a universal upper bound for the partial orderings.

Lemma 26. *OSA ⩾ EER.*

Proof. As in the other lemmas, we show that OSA dominates EER in the partial ordering by considering an equivalent OSA constraint for an arbitrary EER constraint. First, consider an EER-r constraint. Let $Cmin[r(X/Y)] = min$ and $Cmax[r(X/Y)] = max$ be relative cardinality constraints on an n-ary relationship set r, where $X = \{A_1, A_2, \ldots, A_p\}$ and $Y = \{B_1, B_2, \ldots, B_q\}$ are disjoint collections of entity sets involved in r. Suppose X is empty; then the EER constraint is vacuously satisfied. Thus, we need not write any OSA constraint. Now suppose Y is empty, but X is not. Since Y is empty, (EER-r) reduces to

$$\forall \langle a_1 \ldots a_p \rangle \in \pi_{A_1 \ldots A_p} r$$
$$(min \leqslant |\{t \in \pi_{A_1 \ldots A_p} r: t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| \leqslant max) .$$

Now, since the projection of a relationship set is itself a set, tuples in the projection occur exactly once. Since min must be greater than 0, the only value for min that can satisfy this formula is 1. Any positive integer can satisfy max. If $min > 1$, the only relationship set that can satisfy the constraint is the empty relationship set. In this case we write the OSA constraint 0:0 on the relational object class corresponding to r. If, however, $min = 1$ then the EER constraint is trivially satisfied by any r, and we need not write any OSA constraint. The preceding discussion assumes $Cmin$ to be optional. If $Cmin$ is not optional and $min = 1$, then the EER constraint forces $\pi_X r$ to consist of the full cross product of the entity sets in X. We write an equivalent OSA constraint by specifying the co-occurrence constraints

$$A_1A_2 \ldots A_{p-1} \text{-}(|A_p|{:}|A_p|)\!\!\rightarrow A_p$$
$$A_1 \ldots A_{p-2}A_p \text{-}(|A_{p-1}|{:}|A_{p-1}|)\!\!\rightarrow A_{p-1}$$
$$\cdots$$
$$A_2A_3 \ldots A_p \text{-}(|A_1|{:}|A_1|)\!\!\rightarrow A_1$$

together with the participation constraint 1:* on each entity set in X.

We must show that the above OSA-co and OSA-pc constraints enforce $\pi_{A_1 \ldots A_p} r = A_1 \times \cdots \times A_p$ as claimed. By (OSA-co), the first co-occurrence constraint listed above yields

$$\forall \langle a_1, \ldots, a_{p-1} \rangle \in \pi_{A_1 \ldots A_{p-1}} r$$
$$(|\{t \in \pi_{A_1 \ldots A_p} r: t(A_1) = a_1 \wedge \cdots \wedge t(A_{p-1}) = a_{p-1}\}| = |A_p|) .$$

By the participation constraints, we know $\pi_{A_i} r = A_i$ for $A_i \in X$; thus, there is at least one tuple in $\pi_{A_1 \ldots A_{p-1}} r$. Thus, to satisfy the first co-occurence constraint above, each tuple in $\pi_{A_1 \ldots A_{p-1}} r$ must associate with $|A_p|$ different A_p-values in $\pi_{A_1 \ldots A_p} r$. Thus, we know that there are at least $|A_p|$ distinct tuples in $\pi_{A_1 \ldots A_{p-2} A_p} r$. Since there are at least $|A_p|$ distinct tuples in $\pi_{A_1 \ldots A_{p-2} A_p} r$, and since the second co-occurrence constraint listed above forces each tuple in $\pi_{A_1 \ldots A_{p-2} A_p} r$ to associate with $|A_{p-1}|$ A_{p-1}-values, there are at least $|A_p||A_{p-1}|$ tuples in $\pi_{A_1 \ldots A_p} r$. Similar consideration of the third co-occurrence constraint above forces at least $|A_p||A_{p-1}||A_{p-2}|$ tuples in $\pi_{A_1 \ldots A_p} r$. Continuing for all p co-occurrence constraints, yields $|\pi_{A_1 \ldots A_p} r| \geq |A_1||A_2| \cdots |A_p|$. Since the projection is a set, we know that $|\pi_{A_1 \ldots A_p} r| = |A_1||A_2| \cdots |A_p|$, which implies $\pi_{A_1 \ldots A_p} r = A_1 \times \cdots \times A_p$.

We now return to the last case for EER-r. If neither X nor Y is empty, we write the co-occurrence constraint X $-(min: max)\rightarrow$ Y. By (OSA-co), substituting for X and Y, this is

$$\forall \langle a_1, \ldots, a_p \rangle \in \pi_{A_1 \ldots A_p} r$$

$$(min \leq |\{t \in \pi_{A_1 \ldots A_p B_1 \ldots B_q} r \colon t(A_1) = a_1 \wedge \cdots \wedge t(A_p) = a_p\}| \leq max)$$

which is (EER-r). If *Cmin* is not optional, we also write the co-occurrence and participation constraints listed above to enforce $\pi_{A_1 \ldots A_p} r = A_1 \times \cdots \times A_p$.

Finally, we show how an EER absolute cardinality constraint can be written using OSA constraints. Let E be an entity set, with $Cmin[E] = min$ and $Cmax[E] = max$. Then we write the OSA-o constraint min: max for E. By (OSA-o), this yields

$$min \leq |E| \leq max$$

which is (EER-a). Similarly, if E is a relationship set, we specify the OSA-o constraint on the relational object class for E. □

Lemma 27. *OSA ≥ OSA-c.*

Proof. Immediate from the definitions. □

Lemma 28. *EER ≥ EER-r.*

Proof. Immediate from the definitions. □

Counter Example 49. EER $\not\geq$ OSA-pc, because given a binary relationship set r involving entity sets A and B, the OSA participation constraint 1:1,3:* on A cannot be expressed by EER. This is because EER does not allow disjoint ranges to be specified.

Counter Example 50. EER $\not\geq$ OSA-co, because given a binary relationship set r involving entity sets A and B, the OSA co-occurrene constraint A $-(1{:}1{,}3{:}*)\rightarrow$ B on r cannot be expressed by EER. Again, this is because EER does not allow disjoint ranges to be specified.

Counter Example 51. EER-r $\not\geq$ EER, because given an entity set E, EER-r cannot constrain the cardinality of E.

We now have the lemmas needed to establish a universal upper bound for the partial orderings shown in *Figs. 13* and *14*.

Theorem 3. *OSA is a universal upper bound for the partial ordering of cardinality constraints for n-ary relationship sets (Fig. 13), with the additional modification that EER dominates EER-r and all models dominated by EER-r.*

Proof. Theorem 1, Lemmas 26–28, and Counter Examples 49–51. □

Theorem 4. *OSA is a universal upper bound for the partial ordering of cardinality constraints for binary relationship sets (Fig. 14), with the additional modification that EER dominates EER-r and all models dominated by EER-r.*

Proof. Theorem 2, Lemmas 26–28, and Counter Examples 49–51. □

5. Discussion

5.1. Observations

We have presented partial orderings of cardinality constraints for both n-ary and binary relationship sets, together with a universal upper bound. These partial orderings reveal some interesting facts.

First, there are three general kinds of relationship-set cardinality constraints; mapping, participation, and co-occurrence constraints. In broad terms, mapping constraints are a generalization of the mathematical notion of functional versus non-functional mappings. Co-occurrence constraints are a generalization of functional dependencies, of which an important subclass is candidate-key constraints. Participation constraints are a generalization of the idea of total versus partial relationship sets. Boundaries between these categories are fuzzy, but we offer the following categorization. Mapping constraints include SAM, SAM-k, ER, SDM-k, OMT-k, BRM-k, XER, and SM. Participation constraints include SBDM, ECR, SDM, BRM, OSA-pc, OMT-m, and (restricted to binary relationship sets) OSA-co. Co-occurrence constraints include OSA-co, Iris, EER-r, and the candidate-key constraints.

Second, we have seen several different approaches to specifying these constraints. For example, OSA divides its constraints into the three broad classes described above, whereas OMT (for binary relationship sets) and EER (for relationship sets in general) provide a considerable degree of power with only one constraint mechanism. In fact, if EER were to allow disjoint ranges, it could serve as a universal upper bound for the relationship-set partial orderings (but not for the combined entity-set and relationship-set partial orderings, because an OSA entity set that is not even connected to a relationship set can affect the cardinality constraints of the relationship set).

Third, ER model constraints are at or near the bottom of the partial orderings; whereas the later models, EER, OMT, and OSA, are at or near the top. This illustrates that over time, semantic models have expanded the class of information that can be captured using cardinality constraints. It also shows that later work has generally been built on prior work.

Fourth, the relative power of cardinality constraints for some data models changes when comparing binary and n-ary results. Perhaps the most striking example is OMT. In the n-ary case, where candidate keys must be used to specify cardinality, OMT is among the least powerful, but in the binary case OMT dominates most of the partial ordering. EER and OSA, on the other hand, consistently dominate most of the partial ordering for both n-ary and binary cases.

Finally, we observe that the binary relationship-set partial ordering, extended to include OSA as a universal upper bound, forms a lattice. This can be verified by checking each pair of elements in the partial ordering to see that they have unique least upper and greatest lower bounds. The universal lower bound in the binary relationship-set case is SAM-k. The extended n-ary partial ordering, however, even with the addition of a universal lower bound, does not form a lattice. For example, in *Fig. 13*, SAM and XER have no least upper bound, and OSA-c and EER-r have no greatest lower bound. Several other combinations are also missing least upper or greatest lower bounds. This leads us to ask the question, what cardinality constraints are we missing? As an example, we could introduce a new cardinality constraint that dominates SAM and XER, but is dominated by OSA and EER-r. The new cardinality constraint would have functional co-occurrence constraints plus the ability to specify total mappings. We could pursue this course until we arrive at a lattice of cardinality constraints for the n-ary case.

5.2. OSA revisited

We have already seen that OSA has powerful cardinality constraints. OSA has additional features that increase its power and expressiveness even further. OSA allows the use of expressions and variables in defining the min_i and max_i for OSA cardinality-constraint ranges. A term of an expression may include constants, variables, and functions. Functions may include ordinary arithmetic operators (e.g. addition or multiplication), or they may be more complex. We have already seen one of the functions that can be used: the entity-set cardinality function. This allowed us to fix a universal upper bound for the partial orderings.

With variables, for example, we can specify that entity sets A and B both have an object-class cardinality constraint x. This constraint does not specify the actual number of entities in either set, but does ensure that $|A| = |B|$. This notion can be combined with arbitrary functions; for example, entity sets A and B could have object-class cardinality constraints of x and $2x + 7$ respectively, which establishes a relative relationship between the cardinalities of sets A and B.

Each variable in an OSA cardinality constraint is either Type 1 or Type 2. A variable is *Type 1* if it is included in the participation constraints for exactly one object class, and is *Type 2* otherwise. The purpose of a Type 1 variable is to specify a particular value of the variable for each object in the object class; each object may have a distinct value associated with a Type 1 variable. A Type 2 variable, on the other hand, is fixed for all objects in an object class.

The example in *Fig. 16* illustrates both Type 1 and Type 2 variables. Here, we assume that there are processes communicating across a computer network, and that a process can only have one active network session. A process may either establish a direct session to another process, or it may enlist the services of a logical link to perform the appropriate session routing and physical link control. If two processes reside on the same node in the network,

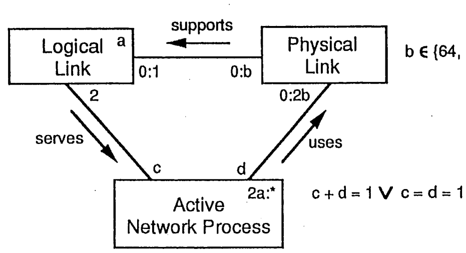

Fig. 16. Variables in OSA cardinality constraints.

then a logical link may bypass a physical link. A physical link may support 64, 128, or 256 simultaneous sessions. To model this, we specify three object classes, *Physical Link*, *Logical Link*, and *Active Network Process*. We assign the variable a to the cardinality of *Logical Link*. Since each logical link serves exactly two processes, the cardinality of *Active Network Process* is bounded below by $2a$. The upper bound is $*$ because a process may establish a direct session, but no information is given in the model regarding the total number of sessions that can be supported by the set of all physical links. Note that a is a Type 2 variable because it is not involved in the specification of any participation constraint. To model the capacity of each physical link, we specify a general constraint $b \in \{64, 128, 256\}$ and then use b in the participation constraints for *Physical Link*. Each physical link can support up to b logical links or up to $2b$ direct uses by active network processes. Because b is determined by the particular physical link involved, we would like to instantiate b separately for each physical link. Since b is a Type 1 variable, we have achieved the desired result. We may, for example, have a physical link supporting 64 logical links and being used by 128 active network processes, and another physical link supporting 256 logical links and being used by 512 active network processes, but we could not have a physical link supporting 24 logical links or one supporting 256 logical links and being used by 64 active network processes. Variables c and d are also Type 1. The general constraint $c + d = 1 \vee c = d = 1$ ensures that an active network process either is using a physical link, or a logical link, or both. We can represent mutual exclusion in a similar fashion, for example with a general constraint $c + d = 1$.

We refer the interested reader to Appendix A of the reference for OSA [9], where a formal definition of OSA is given. There, the meaning of Type 1 and Type 2 variables is formally integrated into the definitions of the OSA cardinality constraints.

5.3. Cardinality constraint extensions

Extensions to the OSA model have also been considered. Sometimes more is known about the participation of an entity in a relationship set than can be expressed with a simple set of integers. For example, the participation of *Doctor* in the relationship set *Doctor has Speciality* may be written as $1{:}*$, but we may know that only a few doctors have more than one specialty. For this case, and for similar cases, a participation constraint of the form *min*: *avg*: *max*, where *avg* is the average of expected participations of entities from the set, may be beneficial. For *Doctor has Specialty*, for example, we might have the cardinality constraint $1{:}1.001{:}*$ on *Doctor*. Similarly, we could tag co-occurrence constraints with an average. We can also generalize this concept by using a probability distribution rather than a simple mean.

6. Conclusion

We have presented formal definitions for cardinality constraints for several semantic data models, including: cardinality constraints for SBDM; 1-1 and 1-many mappings for the ER model; cardinality and dependency constraints for SM; mapping and candidate-key constraints for SAM; cardinality constraints for SDM; cardinality, identifier, total, and uniqueness constraints for BRM; relative cardinality constraints for the EER model; participation constraints for the ECR model; 1-mappings and many-mappings along with mandatory and optional constraints for the XER model; participation constraints for the Iris model; multiplicity and candidate-key constraints for the OMT model; and participation and co-occurrence constraints for the OSA model.

Using these formal definitions, we have developed partial orderings showing the relative power of the relationship-set cardinality constraints. From Theorem 1 we are able to conclude that for n-ary relationship sets, no model's relationship-set cardinality constraints dominates or is subsumed by all others. OSA's participation and co-occurrence constraints together and EER's relative cardinality constraints appear strongest because together they dominate all other models' cardinality constraints. Neither of these strongest constraints, however, dominates the other. ECR participation constraints, SAM mapping and candidate-key constraints, and ER mappings appear weakest because none dominates any of the other models' cardinality constraints. From Theorem 2 we are also able to conclude that for binary relationship sets, no model's cardinality constraints dominates or is subsumed by all others. Candidate-key constraints are the only cardinality constraints that do not dominate any others for binary relationship sets. EER relative cardinality constraints, and OMT multiplicity constraints which are equivalent to OSA participation constraints, are the dominant cardinality constraints for binary relationship sets, but neither dominates the other.

For EER and OSA, we also presented entity-set cardinality constraints. EER has an absolute cardinality constraint that applies to entity sets and relationships sets. OSA has an object-class cardinality constraint and also allows the use of object-class cardinalities in the specification of all its cardinality constraints. We demonstrated how to write EER absolute and relative cardinality constraints using OSA constraints, and thus showed that OSA has the most powerful set of combined relationship-set and entity-set cardinality constraints. This led to the universal upper bound of Theorems 3 and 4. With the universal upper bound of Theorem 4, it can be shown that the resulting partial ordering for cardinality constraints on binary relationship sets is a lattice.

We presented a number of observations about the partial orderings and discussed additional features of OSA, including the use of variables and expressions in cardinality-constraint specification. We also suggested a possible extension to OSA cardinality constraints including the use of averages and probability distributions, which would be useful for implementation purposes.

Possibilities for future work include discovering new cardinality constraints sufficient to expand our n-ary partial ordering so that it becomes a lattice. Also, it is likely that more powerful cardinality-constraint ideas could be discovered as a result of this formal comparison of existing cardinality constraints for semantic data models.

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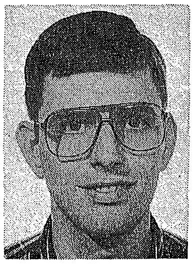

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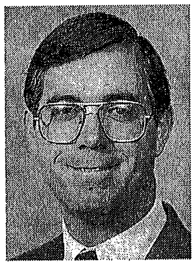

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